On approximate formulas for low Prandtl number heat transfer in laminar wedge flows

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The heat transfer in Falkner-Skan similarity boundary layers is considered. Shortcomings of some approximate formulas for the local Nusselt number are pointed out and discussed. The importance of an accurate representation of the flow field in the outer region of the thermal boundary layer is emphasized.

Keywords: wall heat transfer; boundary layers; wedges; approximations

Introduction

The exact solution of the thermal energy equation for flow in a boundary layer of the Falkner-Skan type was derived by Fage and Falkner[1] some 50 years ago. However, this solution is given in integral form, and simple algebraic approximations may therefore still be of some relevance in engineering applications and for design purposes. Furthermore, in dealing with more complex heat transfer problems, one may benefit from the insight gained through the simpler approximate analyses.

The objective of the present paper is to consider some asymptotic formulas for the local Nusselt number when the Prandtl number is small, i.e., appreciably less than unity. Although the Prandtl number of most liquids is greater than unity, liquid metals which are frequently encountered in nuclear reactor technology[2,3] have low Prandtl numbers, typically about 0.01. Moreover, the extreme Prandtl number results (i.e., high and low Prandtl number asymptotes) are interesting in themselves and provide useful approximations to the exact solution, even in the range $\mathrm{Pr} \approx 1$.

An interesting approach to the low Prandtl number regime is the two-region formulation considered recently by Chen[4], in which the thermal boundary layer is divided into two distinct regions. The region next to the wall extends over the velocity boundary layer, and the outer region extends over the inviscid flow domain. In the heat transfer literature, however, there seems to be some confusion about the various approximations involved in the asymptotic approaches. The intention of the paper is therefore to elucidate this issue.

First, one of the approximate formulas for the temperature gradient at the wall is improperly given in the textbook by Evans[5]. This appears not to have been realized before, and the corrected expression will be provided in this paper.

Second, the approximations used by Chen[4] are considered. It is pointed out that the temperature derived in the outer region is inconsistent with the approximations employed for the velocity field. However, it is easily verified that the resulting formula for the local Nusselt number incidentally recovers a very accurate and useful result derived by Evans[5].

The thermal energy equation and its exact solution

The calculation of the heat transfer rate in a two-dimensional, steady, laminar boundary layer requires the solution of the thermal energy equation

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=K\frac{\partial^2 T}{\partial y^2} \tag{1}$$

The fluid properties have been considered uniform throughout the flow, and internal dissipation of energy is assumed negligible. In accordance with classical boundary layer arguments, longitudinal heat conduction has been neglected in Equation 1. The accuracy of this approximation for high Peclet number flow of liquid metals has been verified by Grosh and Cess[2].

Exact, or approximate, knowledge about the velocity field is required in order to derive the temperature field $T(x, y)$ from Equation 1. Provided the inviscid flow belongs to the Falkner-Skan family, in which the velocity distribution obeys the relation

$$U(x) \sim x^m \tag{2}$$

the velocity components in the viscous boundary layer can be obtained from the dimensionless stream function $f(\eta)$ according to

$$u=\frac{\partial\psi}{\partial y}, \qquad v=-\frac{\partial\psi}{\partial x} \tag{3}$$

$$\psi=f\sqrt{\frac{2}{m+1}\,U\nu x} \tag{4}$$

$$\eta=y\sqrt{\frac{m+1}{2}\frac{U}{\nu x}} \tag{5}$$

Here, Equations 4 and 5 represent the well-known transformation due to Hartree[6], which reduces the boundary layer momentum equation to an ordinary differential equation. Correspondingly, the thermal energy equation (1) transforms into the ordinary equation

$$\theta''+\sigma f\theta'=0 \tag{6}$$

where $\theta=1-(T-T_\infty)/(T_w-T_\infty)$ is a dimensionless temperature and $\sigma=\nu/K$ is the Prandtl number. The prime signifies differentiation with respect to η.

The complete solution to the energy equation (6) subject to the boundary conditions $\theta(0)=0$ and $\theta\to 1$ as $\eta\to\infty$ is found elsewhere (e.g., Evans[5]). Of particular significance is the heat transfer rate, for which the Nusselt number is a convenient measure. For the heat transfer in boundary layers driven by a free stream of the particular form in Equation 2, the local Nusselt number becomes

$$\mathrm{Nu}_x = \theta'_w \sqrt{\frac{m+1}{2}\,\mathrm{Re}_x} \tag{7}$$

$$\frac{1}{\theta'_w} = \int_0^\infty \exp\left[-\sigma \int_0^t f(\eta)\,d\eta\right] dt \tag{8}$$

where $\mathrm{Re}_x = Ux/\nu$ is the local Reynolds number and $\theta'_w \equiv \theta'(0)$ is the wall gradient of the dimensionless temperature field. It is important to emphasize that the result in Equations 7 and 8 is exact only if the variation of the stream function f is exactly represented by a solution of the momentum equation.

Approximate solutions for small Prandtl number

The low Prandtl number regime is characterized by a thermal boundary layer which extends far into the inviscid mainstream, so most of the thermal resistance is contributed by the region outside the velocity boundary layer. In Chapter 7.3 of Evans[5] the flow is fairly well approximated by $f = \eta - \eta^*$, which represents an inviscid flow field displaced a distance η^* away from the wall. Here, η^* denotes the nondimensional displacement thickness defined as

$$\eta^* \equiv \int_0^\infty (1-f')\,d\eta = \frac{\delta^*}{x}\sqrt{\frac{m+1}{2}\,\mathrm{Re}_x} \tag{9}$$

where δ^* is the conventional displacement thickness of the velocity boundary layer.

With $f = \eta - \eta^*$, the integral in Equation 8 can be evaluated, and the reciprocal of the wall gradient becomes

$$\frac{1}{\theta'_w} = \left(\frac{\pi}{2\sigma}\right)^{1/2} \left\{1 + \mathrm{erf}\left[\left(\frac{\sigma}{2}\right)^{1/2} \eta^*\right]\right\} \exp\left(\frac{\sigma}{2}\eta^{*2}\right) \tag{10}$$

In the corresponding solution by Evans[5], the exponential factor has unfortunately been left out; the correct solution, Equation 10, was given earlier by Merk[7]. If the displacing effect of the velocity boundary layer is completely neglected—i.e., if $\eta^* = 0$—Equation 10 reduces to

$$1/\theta'_w = (\pi/2\sigma)^{1/2} \tag{11}$$

in accordance with the solution given by Evans[5] for a stream function approximated as $f = \eta$.

For later use, it is instructive to expand Equation 10 in powers of the parameter $(\sigma/2)^{1/2}\eta^*$. The leading terms in the resulting series become

$$1/\theta'_w = (\pi/2\sigma)^{1/2} + \eta^* + (\pi\sigma/8)^{1/2}\eta^{*2} + \cdots \tag{12}$$

which is a reasonable approximation of Equation 10 in the limit $\sigma \to 0$.

Chen[4] suggested that improved accuracy could be obtained by modeling the heat transfer in the viscous boundary layer and in the inviscid flow separately. Accordingly, the temperature field was divided in two distinct regions, and the displacement thickness δ^* defines the boundary between them.

Chen neglected the convective terms in Equation 1 in the inner region $0 \leqslant y < \delta^*$, and assumed a velocity field given by $\psi = yU$ as a reasonable approximation for $\delta^* \leqslant y < \infty$. By integrating the energy equation separately for the two regions, he obtained a linear variation of the temperature in the inner region, which could be matched with the outer region solution. By this method, Chen was able to derive an approximate formula for the local Nusselt number which compared favorably with the exact solution for Prandtl numbers less than 0.1. In fact, he demonstrated that predictions by the approximate formula were within 3.2% of the exact results for all values of the free-stream parameter m.

Unfortunately, Chen's solution for the temperature variation in the outer region is not consistent with the resulting energy equation in that region, and the Nusselt number formula obtained is accordingly inconsistent with the approximations made. The corrected solution, however, gives the reciprocal wall gradient as

$$\frac{1}{\theta'_w} = \eta^* + \left(\frac{\pi}{2\sigma}\right)^{1/2} \left\{1 - \mathrm{erf}\left[\left(\frac{\sigma}{2}\right)^{1/2} \eta^*\right]\right\} \exp\left(\frac{\sigma}{2}\eta^{*2}\right) \tag{13}$$

which, expanded in powers of $(\sigma/2)^{1/2}\eta^*$, becomes

$$1/\theta'_w = (\pi/2\sigma)^{1/2} + (\pi\sigma/8)^{1/2}\eta^{*2} + \cdots \tag{14}$$

for small values of σ.

The assertion by Chen that "a new approach based on a two-region model is proposed" is questionable since Evans introduced a two-region approximation in his excellent textbook[5]. More specifically, he assumed a stream function f which vanished for $\eta \leqslant \eta^*$ and was given by $f = \eta - \eta^*$ in the outer region $\eta \geqslant \eta^*$. This is equivalent to assuming a step function for the velocity profile f' consisting of a step of unit height located at $\eta = \eta^*$. Substituting this approximation for f into the integral-form solution in Equations 7 and 8 and

Notation

- f Dimensionless stream function
- K Thermal diffusivity
- m Exponent in inviscid velocity distribution
- Nu_x Local Nusselt number
- Re_x Local Reynolds number, Ux/ν
- T Temperature
- u, v Velocity components in the x, y coordinate directions
- U Free-stream velocity
- x Streamwise coordinate
- y Cross-stream coordinate

Greek symbols

- δ^* Velocity boundary layer displacement thickness
- η Dimensionless coordinate
- θ Dimensionless temperature, $1-(T-T_\infty)/(T_w-T_\infty)$
- ν Kinematic viscosity
- σ Prandtl number, ν/K
- ψ Stream function

Subscripts

- w Condition at the wall
- ∞ Condition outside the boundary layers

Superscript

- $*$ Denotes a conventional displacement thickness

Function

$$\mathrm{erf}\,x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\,dz = -\mathrm{erf}(-x)$$

integrating, we get

$$1/\theta'_w = (\pi/2\sigma)^{1/2} + \eta^* \quad (15)$$

Even though this result was derived from the exact solution, Equation 8, Equation 15 is evidently an approximate formula, since the exact variation of f was roughly accounted for by the two-region approximation.

Discussion

It is interesting to compare the assumptions involved in the two-region approaches of Chen[4] and Evans[5] in Table 1. The inner region approximation $f=0$ is obviously equivalent to the neglect of the convective terms in Equation 1. In the outer region, however, the stream function $f=\eta-\eta^*$, as assumed by Evans, is clearly different from the velocity field considered by Chen; the latter corresponds to $f=\eta$. Nevertheless, it is remarkable that the incorrectly calculated expression for the Nusselt number due to Chen is equivalent to Equation 15 obtained by Evans, and, furthermore, that this equivalence went unnoticed through the reviewing procedure. It can readily be verified, however, that the temperature variation as calculated by Chen for the outer region corresponds to the velocity field represented by the stream function $f=\eta-\eta^*$ rather than $f=\eta$.

The various formulas for the reciprocal wall gradient are derived from four distinct approximations of the velocity field. Figure 1 displays the corresponding variation of the stream function as compared with an exact solution for f. Here, it should be kept in mind that Equations 12 and 14 are nothing else but power expansions of Equations 10 and 13, respectively.

Now, examination of the various formulas 10–15 reveals that for small Prandtl numbers

(i) Equations 10 and 12 approach Equation 15.
(ii) Equations 13 and 14 approach Equation 11.

It can therefore be concluded that the assumption made for the velocity field in the outer region is important, and the influence of the inner region approximation is less significant for the smaller values of σ. With reference to Figure 1, it may thus be anticipated that Equations 10 and 15 are superior to Equations 11 and 13, simply because they represent a better approximation to the exact stream function in the outer region of the thermal boundary layer. This is furthermore confirmed by the comparison of the various formulas in Table 2, which also shows that the two-region approach by Evans[5], i.e., Equation 15, is superior to the other formulas.

Table 1 Two-region approximations for the velocity field

Variable	Inner region ($0 \leq \eta < \eta^*$)	Outer region ($\eta^* \leq \eta < \infty$)	
		Chen[4]	Evans[5] eq. 7.5
ψ	0	yU	$(y-\delta^*)U$
f	0	η	$\eta-\eta^*$
u	0	U	U
v	0	$-y(dU/dx)$	$-(y-\delta^*)\,dU/dx+U(d\delta^*/dx)$

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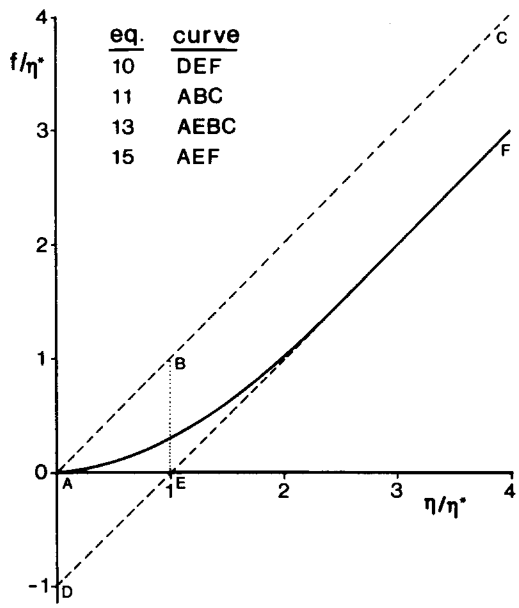

Figure 1 Various approximations for f (broken lines) as compared with the exact solution (solid line)

Table 2 Percentage error in Nu_x as calculated by the approximate formulas 10–15. Flat-plate boundary layer ($m=0$) with $\eta^*=1.21678$

σ	$Nu_x\left(Re_x\frac{m+1}{2}\right)^{-1/2}$ Exact (Evans[5])	Error, 100% $[Nu_x-(Nu_x)_{exact}]/(Nu_x)_{exact}$ Eq. 10	Eq. 11	Eq. 12	Eq. 13	Eq. 14	Eq. 15
10^{-4}	0.00790224	−0.0089	0.97	−0.0087	0.96	0.96	−0.0014
10^{-3}	0.0244880	−0.11	3.0	−0.11	3.0	3.0	−0.034
10^{-2}	0.0729570	−1.0	9.4	−0.98	8.6	8.6	−0.31
10^{-1}	0.198031	−9.0	27.	−7.7	20.	19.	−2.5
1	0.469600	−54.	70.	−37.	18.	−2.4	−14.